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FERMI–WALKER TRANSPORTS OVER SPACES WITH AFFINE CONNECTIONS AND METRICS*

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The notion of Fermi–Walker transport is generalized for the case of differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and metrics [$(\overline{L}_n, g)$-spaces]. The existence of such type of transport over $(\overline{L}_n, g)$-spaces allows the determination of a proper nonrotating accelerated observer's frame of reference if a $(\overline{L}_n, g)$-space is used as a model of the space-time.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Переносы Ферми–Уолкера в пространствах с аффинными связностями и метрикой

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Рассмотрен перенос Ферми–Уолкера в V_n- и U_n-пространствах. Найдено обобщение для пространств с аффинными связностями и метрикой [$(\overline{L}_n, g)$-пространств]. Обобщение не является однозначным и зависит от структуры ковариантного антисимметричного тензора второго ранга в конструкции переноса Ферми–Уолкера. Существование такого типа переноса в $(\overline{L}_n, g)$-пространствах позволяет определить собственную невращающуюся систему ускоренного наблюдателя, если $(\overline{L}_n, g)$-пространство рассматривается как модель пространства-времени.

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1. Introduction

The last time evolution of mathematical models for describing the gravitational interaction on a classical level shows a tendency to generalizations using spaces with affine connection and metric [1], [2], which can be also generalized using the freedom of the different-geometric preconditions. The fact that an affine connection, which in a point or over a curve in Riemannian spaces can vanish (principle of equivalence in the Einstein theory of gravitation (ETG)) and can also vanish under a special choice of the basic system in a space with affine connection and metric [3], [4], [5], shows that the equivalence

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principle in the ETG is only a corollary of the mathematical apparatus used in this theory. Therefore, *every differentiable manifold with affine connection and metric can be used as a model for space-time in which the equivalence principle holds.* Even if the manifold has two different (not only by sign) connections for tangent and co-tangent vector fields the principle of equivalence is fulfilled for the one of the two types of vector fields [6].

The notion of a nonrotating observer's frame of reference is related in the Einstein theory of gravitation to a special type of a transport of contravariant vector fields along a nonisotropic (non-null) contravariant vector field u considered as a tangent vector field to the trajectory of the observer. This special type of transport, called Fermi–Walker transport [7], does not change the length of a covariant nonisotropic vector fields.

A *proper frame of reference* can also be interpreted as *a congruence of trajectories (a set of nonintersecting curves) of particles moving with an acceleration in such a way that the distance between all particles with equal (one and the same) proper time and the angles between the vectors connecting them from a given basic trajectory do not change along their trajectories.*

On the other side, the Fermi–Walker transport is related to the so-called relative transport (called also Fermi derivative) as well as to the relative velocity and the connected with it notions of shear, rotation and expansion [8].

1.1. *Fermi–Walker Transport in V_n- and U_n-Spaces*. On the grounds of the heuristic considerations of the properties of a proper frame of reference in V_n-spaces ($n=4$) the Fermi–Walker transport $\nabla_u \xi = {}^F\nabla_u \xi$ has been proposed in the form [9], [10]

$$ {}^F\nabla_u \xi = \frac{1}{e} [a \otimes g(u) - u \otimes g(a)](\xi) = \bar{g}\, [{}^F\tilde{\omega}(\xi)] \,, \tag{1} $$

where $e = g(u, u) = \text{const.} \neq 0$, $\nabla_u u = a$, $\xi \in T(M)$.

The difference $\nabla_u \xi = -{}^F\nabla_u \xi = {}^r\nabla_u \xi$ determines the s.c. relative transport ${}^r\nabla_u \xi$ called also Fermi derivative. Therefore, the Fermi–Walker transport can also be defined by the condition ${}^r\nabla_u \xi = 0$.

The vector field u fulfills automatically the condition for a Fermi–Walker transport. At the same time

$$ {}^F\tilde{\omega}(\xi) = \frac{1}{e} [g(u, \xi) g(a) - g(a, \xi) g(u)] \,. \tag{2} $$

It has to be stressed that the properties of the Fermi–Walker transport (FWT) are not independent of the explicit structure of ${}^F\tilde{\omega}$. The only important property of ${}^F\tilde{\omega}$ leading to its characteristics is that ${}^F\tilde{\omega}$ is a covariant antisymmetric tensor field of second rank $[{}^F\tilde{\omega} \in \Lambda^2(M)]$.

A comparison of the structure of the Fermi–Walker transport with the structure of the rotation velocity tensor ω as an element of the relative velocity ${}_{\text{rel}}v$ in V_n-spaces shows that ω contains the term ${}^F\tilde{\omega}$ which determines the FWT

$$\omega = h_u(s)h_u = g(s)g - \frac{1}{e}\{g(u) \otimes [g(u)](s)g - [g(u)](s)g \otimes g(u)\} =$$
$$= g(s)g - \frac{1}{2e}[g(u) \otimes g(a) - g(a) \otimes g(u)] = \omega_0 + \frac{1}{2}{}^F\tilde{\omega}, \tag{3}$$

where $\omega_0 = g(s)g$, $[g(u)](s)g = \frac{1}{2}g(a)$, $g(u, u) = e = \text{const.} \neq 0$. ω_0 is interpreted as a space rotation [9].

Instead of the form for ${}^F\tilde{\omega}$ in V_u-spaces, we can use the form of ${}^F\overline{\omega}$ for nonisotropic vector field u in U_n-spaces [8]. Then, for U_n-spaces we will have

$$^F\overline{\omega} = \frac{2}{e}\{g(u) \otimes [g(u)](s)g - [g(u)](s)g \otimes g(u)\}. \tag{4}$$

The relation $\nabla_u\xi = {}^F\overline{\nabla}_u\xi = \overline{g}[{}^F\overline{\omega}(\xi)]$ determines the FWT in U_n-spaces. It can also be expressed by means of the relative transport (the Fermi derivative) ${}^r\nabla_u\xi = \nabla_u\xi - {}^F\overline{\nabla}_u\xi$ using the condition ${}^r\nabla_u\xi = 0$.

In the case $e = \text{const.} \neq 0$ and $T(\xi, \eta) = 0$ for $\forall\, \xi, \eta \in T(M)$ $(T_{ij}^{\ k} = 0)$, the FWT passes to that in V_n-spaces.

One can summarize the properties of the FWT in V_n- and U_n-spaces in the following definitions depending on the physical interpretation of the Fermi–Walker transport.

Definition 1. *The Fermi–Walker transport* is a special type of transport (along a non-isotropic contravariant vector field) preserving the length of the transported contravariant nonisotropic vector fields and the angle between them.

The interpretation of the Fermi–Walker transport as a transport preserving the length of a set of orthogonal to each other nonisotropic vector fields and the right angles between them or as a transport preserving the proper nonrotating frame of reference of an observer moving with acceleration $\nabla_u u = a \neq 0$ in the space-time, allows a weaker definition of the Fermi–Walker transport [9].

Definition 2. *The Fermi–Walker transport* is a special type of transport (along a nonisotropic contravariant vector field) preserving the length of a set of orthogonal to each other nonisotropic vector fields and the right angles between them.

The last definition is restricted on orthogonal nonisotropic vector fields only.

The two different to each other definitions allow different generalization of the Fermi–Walker transport over differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and metrics [$(\overline{L}_n, g)$-spaces].

2. Generalized Fermi–Walker Transports over $(\overline{L}_n, g)$-Spaces

The nonmetricity q_u $[\nabla_u g = q_u \neq 0$ for $\forall\, u \in T(M)]$ over differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and metrics

[$(\overline{L}_n, g)$-spaces] [6] does not allow the direct application of the explicit forms of the Fermi–Walker transport in V_n- and U_n-spaces. The rate of change of the length of a nonisotropic contravariant vector field and the rate of change of the cosine of the angle between two nonisotropic vector fields over a $(\overline{L}_n, g)$-space are given by the expressions

$$ul_\xi = \pm \frac{1}{2l_\xi} [(\nabla_u g)(\xi, \xi) + 2g(\nabla_u \xi, \xi)], \; l_\xi \neq 0, \tag{5}$$

$$u[\cos(\xi, \eta)] = \frac{1}{l_\xi l_\eta} [(\nabla_u g)(\xi, \eta) + g(\nabla_u \xi, \eta) + g(\xi, \nabla_u \eta)] - $$

$$- \left[\frac{1}{l_\xi}(ul_\xi) + \frac{1}{l_\eta}(ul_\eta) \right] \cos(\xi, \eta). \tag{6}$$

2.1. *Definition and Properties of a Generalized Fermi–Walker Transport.* On the ground of the basic properties of a Fermi–Walker transport (FWT) (s. the first definition of FWT) one can introduce the following definition

Definition 3. A generalized Fermi–Walker transport $\nabla_u \xi = {}^F\nabla_u \xi$ along a nonisotropic vector field u is a transport of the type

$$\nabla_u \xi = \overline{g}({}^F\omega - \frac{1}{2}\nabla_u g)(\xi) = {}^F\nabla_u \xi = \overline{g}\,[{}^F\omega(\xi)] - \frac{1}{2}\overline{g}(\nabla_u g)(\xi), \tag{7}$$

where ${}^F\omega = {}^F\omega_{ij}\, dx^i \wedge dx^j = {}^F\omega_{\alpha\beta} e^\alpha \wedge e^\beta$, ${}^F\omega \in \Lambda^2(M)$, $g(u, u) = e \neq 0$, $u, \xi \in T(M)$.

From the definition of FWT and the expression for ul_ξ and $u[\cos(\xi, \eta)]$ the properties of this type of transport follow.

Proposition 1. *The generalized Fermi–Walker transport along a nonisotropic vector field u ($l_u \neq 0$) preserves the length of energy arbitrary given nonisotropic contravariant vector field ξ.*

The proof follows from the expression for ul_ξ and the definition for ${}^F\nabla_u \xi$. Therefore, for the vectors ξ, η and u the relations $ul_\xi = 0$, $ul_\eta = 0$, $ul_u = 0$, [$\xi, \eta, u \in T(M)$], are fulfilled.

Proposition 2. *The generalized Fermi–Walker transport along a nonisotropic vector field u ($l_u \neq 0$) preserves the angle between two contravariant nonisotropic vector fields ξ and η, i.e., $u[\cos(\xi, \eta)] = 0$ for $u, \xi, \eta \in T(M)$.*

The proof follows from the expression for the rate of change $u[\cos(\xi, \eta)]$ of $\cos(\xi, \eta)$ along u and the proposition 1. For $\eta = u$ the condition $u[\cos(\xi, \eta)] = 0$ also follows.

The generalized Fermi–Walker transport of u along u leads to the condition

$$\nabla_u u = a = {}^F\nabla_u u = \overline{g}\,[{}^F\omega(u)] - \frac{1}{2}\overline{g}(\nabla_u g)(u). \tag{8}$$

Different antisymmetric covariant tensor fields of second rank ${}^{F}\omega \in \Lambda^{2}(M)$ induce different generalized Fermi–Walker transports. The same statement is also valid for different types of metric transport $\nabla_u g$.

One can consider a generalized FWT along u as a special case of a relative transport ${}^{r}\nabla_u \xi$ along u defined as

$$ {}^{r}\nabla_u \xi = \nabla_u \xi - {}^{F}\nabla_u \xi = \left[\nabla_u - \bar{g}({}^{F}\omega - \frac{1}{2}\nabla_u g \right](\xi), \tag{9} $$

with ${}^{r}\nabla_u = \nabla_u - {}^{F}\nabla_u = \nabla_u - \bar{g}({}^{F}\omega - \frac{1}{2}\nabla_u g)$.

The condition ${}^{r}\nabla_u = 0$, $u, \xi \in T(M)$, determines a generalized FWT. ${}^{r}\nabla_u \xi$ can also be called generalized Fermi derivative of a contravariant vector field ξ along a contravariant vector field u over a $(\bar{L}_n, g)$-space.

3. Special Cases of Generalized Fermi–Walker Transports

Let we now specify the construction of the antisymmetric covariant tensor field ${}^{F}\omega$ used for the determination of a generalized Fermi–Walker transport.

3.1. *Canonical Fermi–Walker Transport.* The canonical FWT can be considered as a simple generalization of the FWT in V_n-spaces to FWT in $(\bar{L}_n, g)$-spaces, where ${}^{F}\omega = {}^{F}\tilde{\omega}$,

$$ \nabla_u \xi = {}^{F}\nabla_u \xi = \bar{g}[{}^{F}\tilde{\omega}(\xi)] - \frac{1}{2}\bar{g}(\nabla_u g)(\xi) = \bar{g}\left({}^{F}\tilde{\omega} - \frac{1}{2}\nabla_u g \right)(\xi), \tag{10} $$

with ${}^{F}\tilde{\omega} = \frac{1}{e}[g(a) \otimes g(u) - g(u) \otimes g(a)]$, ${}^{F}\tilde{\omega}(u) = g(a) - \frac{1}{e} g(u, a) g(u) = h_u(a)$, $g(u, a) = g(a, u) = \frac{1}{2}[ue - (\nabla_u g)(u, u)]$, ${}^{F}\nabla_u = \bar{g}\left({}^{F}\tilde{\omega} - \frac{1}{2}\nabla_u g \right)$.

For this type of Fermi–Walker transport the relations are valid:

$$ ul_\xi = 0, \; ul_\eta = 0, \; ul_u = 0, \; \xi, \eta, u \in T(M), \; g(u, u) = e \neq 0, \tag{11} $$

$$ \nabla_u u = a = {}^{F}\nabla_u u = \bar{g}[h_u(a)] - \frac{1}{2}\bar{g}(\nabla_u g)(u). \tag{12} $$

From the last expression, it follows that u does not fulfil automatically the conditions for the canonical Fermi–Walker transport.

For the cosine of the angle between two contravariant nonisotropic vector fields we obtain

$$ u[\cos(\xi, \eta)] = 0, \; u[\cos(\xi, u)] = 0, \; u[\cos(\eta, u)] = 0, \; \xi, \eta, u \in T(M). \tag{13} $$

3.2. *Fermi–Walker Transport Related to the Rotation Velocity.* If we close the antisymmetric covariant tensor ω from the representation of the relative velocity by means of shear, rotation and expansion [11], then we can define a Fermi–Walker transport of the type

$$\nabla_u \xi = {}^F\nabla_u \xi = \bar{g}[\omega(\xi)] - \frac{1}{2}\bar{g}(\nabla_u g)(\xi) = \bar{g}(\omega - \frac{1}{2}\nabla_u g)(\xi), \tag{14}$$

where

$${}^F\nabla_u = \bar{g}\left(\omega - \frac{1}{2}\nabla_u g\right), \quad \omega = h_u(s-q)h_u. \tag{15}$$

For the contravariant nonisotropic vector field u the Fermi–Walker transport has the form

$$\nabla_u u = \bar{g}\left(\omega - \frac{1}{2}\nabla_u g\right)(u) = \bar{g}(\omega)(u) - \frac{1}{2}\bar{g}(\nabla_u g)(u) = -\frac{1}{2}\bar{g}(\nabla_u g)(u) = -\bar{g}[\nabla_u(g(u))], \tag{16}$$

since $\bar{g}(\omega)(u) = 0$, $(\nabla_u g)(u) = \nabla_u[g(u)] - g(\nabla_u u)$.

4. Fermi–Walker Transport for Orthogonal Vector Fields

The use of the second definition of Fermi–Walker transport concerning the preservation of the length of orthogonal to each other vector fields only and the right angle between them leads to another form of the generalized Fermi–Walker transport.

Definition 4. $\nabla_u \xi = {}^F\overline{\nabla}_u \xi$ with

$$\nabla_u \xi = {}^F\overline{\nabla}_u \xi = \frac{l_{u\xi}}{e}\bar{g}[h_u(a)] + \bar{g}(\omega)(\xi) - \frac{1}{2}\bar{g}(\nabla_u g)(\xi), \tag{17}$$

where $l_{u\xi} g(u, \xi) = l$, $e = g(u, u) \neq 0$, $\xi, u \in T(M)$,

$${}^F\overline{\nabla}_u = \frac{l}{e}\bar{g}[h_u(a)] \otimes g(u) + \bar{g}(\omega) - \frac{1}{2}\bar{g}(\nabla_u g), \tag{18}$$

is called Fermi–Walker transport for orthogonal vector fields. It is obvious that if the vector field ξ is orthogonal to u ($l_{\xi u} = l = 0$), then $\nabla_u \xi = {}^F\overline{\nabla}_u \xi = {}^F\nabla_{u\xi}$ is the FWT related to the relative velocity.

If $\nabla_u u = {}^F\overline{\nabla}_u u$, then $\nabla_u u = a$ has to fulfil the equation $\nabla_u u = a = \bar{g}[h_u(a)] - \frac{1}{2}\bar{g}(\nabla_u g)(u)$.

By the use of the definition of the length of a contravariant nonisotropic vector field and the definition of the cosine between two nonisotropic contravariant vector fields together with the expressions for their rate of change along a nonisotropic vector field u one can prove the propositions

Proposition 3. *The Fermi–Walker transport* $\nabla_u \xi = {}^F\overline{\nabla}_u \xi$ *preserves the length of an orthogonal to the nonisotropic vector field u contravariant vector field* ξ *along u.*

Proposition 4. *The Fermi–Walker transport* $\nabla_u \xi = {}^F\overline{\nabla}_u \xi$ *preserves the angle between two, orthogonal to the nonisotropic vector field u, contravariant nonisotropic vector fields* ξ *and* η.

If the vector fields ξ and η are orthogonal to u vector fields, i.e., if $l_{u\xi} = g(u, \xi) = 0$ and $l_{u\eta} = g(u, \eta) = 0$, then $u[\cos(\xi, \eta)] = 0$.

The rate of change of the right angle between the vector field u and a contravariant, nonisotropic, and orthogonal to u vector field ξ follows from the expression (6) for $\eta = u$ in the form

$$u[\cos(\xi, u)] = \frac{1}{l_\xi l_u} h_u(a, \xi) = \frac{1}{l_\xi l_u} g(a, \xi), \tag{19}$$

where $l_{uu} = g(u, u) = e = l_u^2$, $l_{u\xi} = l = 0 = g(u, \xi)$, $l_\xi \neq 0$, $h_u(a, u) = 0$.

Therefore, if the right angle between u and ξ has not to change along u, then ξ has to be orthogonal to the acceleration a, i.e., if $u[\cos(\xi, u)] = 0$, then $g(a, \xi) = 0$ and vice versa. On the other side, the angle between two orthogonal to u vector fields ξ and η could be different from the right angle.

Remark. For an orthogonal to u contravariant and nonisotropic vector field ξ: ${}^F\overline{\nabla}_u\xi = {}^F\nabla_u\xi = \overline{g}\left(\omega - \frac{1}{2}\nabla_u g\right)(\xi)$.

5. Conclusion

The existence of Fermi–Walker transports over $(\overline{L}_n, g)$-spaces allows the determination of a proper nonrotating, accelerated observer's frame of reference for this type of spaces if used as models of the space-time. Therefore, the $(\overline{L}_n, g)$-spaces could have meaningful applications in models of physical systems and their interactions and especially in models concerning the gravitational interaction. The different types of Fermi–Walker transports could also be used for description of the motion of moving in external fields particles with equal proper times and constant distance and angles between them considered from the trajectory of one of the particles.

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